Cyclical Behaviour in Early Universe Cosmologies

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Abstract

We study early universe cosmologies derived from a scalar–tensor action containing cosmological constant terms and massless fields. The governing equations can be written as a dynamical system which contains no past or future asymptotic equilibrium states (i.e. no sources nor sinks). This leads to dynamics with very interesting mathematical behaviour such as the existence of heteroclinic cycles. The corresponding cosmologies have novel characteristics, including cyclical and bouncing behaviour possibly indicating chaos. We discuss the connection between these early universe cosmologies and those derived from the low–energy string effective action.

1 Introduction

In this paper we consider the qualitative dynamics of a class of spatially flat, scalar–tensor cosmological models derived from the action

$$S=\int d^4x\sqrt{-g}\left\{e^{-\Phi}\left[R+(\nabla\Phi)^2-\frac{1}{2}e^{2\Phi}\left(\nabla\sigma\right)^2-2\Lambda\right]-\Lambda_{\rm M}\right\} \tag{1}$$

where R is the Ricci curvature scalar of the space–time with metric $g_{\mu\nu}$, $g\equiv \det g_{\mu\nu}$, $\{\Lambda,\Lambda_{\rm M}\}$ are constants and $\{\Phi,\sigma\}$ represent scalar fields. The dynamics of these cosmological models has very interesting mathematical properties. In particular, there are no asymptotically attracting equilibrium states in the phase space and this may lead to important physical consequences.

The form of action (1) can be partially motivated from string theory, which is the most promising candidate for a unified theory of the fundamental interactions [1, 2]. When $\Lambda_{\rm M}$ vanishes, Eq. (1) represents the truncated effective four–dimensional action of the Neveu–Schwarz/Neveu–Schwarz (NS–NS) sector of the theory [1]. The scalar field, Φ, represents the dilaton field and the axion field, σ, is the Poincaré dual of the antisymmetric two–form potential. The constant, Λ, may be interpreted in terms of the central charge deficit of the string theory and can be negative. The evolution of the very early universe immediately below the string scale may have been determined by an effective action of this form. The dynamics of the spatially flat and homogeneous cosmologies in the case $\Lambda_{\rm M}=0$ was presented in Ref. [3]. One of the main purposes of the present work is to determine the effects of introducing a non–trivial cosmological constant, $\Lambda_{\rm M}$, that does not couple directly to the dilaton field. Such a term represents a vacuum energy contribution to the energy–momentum tensor. A discussion of the spatially flat and homogeneous cosmologies in the case $\Lambda=0$ was presented in Ref. [4], and a partial analysis of the dynamics with a non-vanishing axion field when both Λ and $\Lambda_{\rm M}$ are non-zero was investigated in [5].

2 Analysis

We assume that the metric corresponds to the spatially flat, Friedmann–Robertson–Walker (FRW) universe: $ds^2=-dt^2+e^{2\alpha(t)}dx_idx^i$. Substituting this *ansatz* into the action (1) and integrating over the spatial variables then yields the reduced action

$$S=\int dt\ \ e^{3\alpha}\left\{e^{-\Phi}\left[6\dot{\alpha}\dot{\Phi}-6\dot{\alpha}^2-\dot{\Phi}^2+\frac{1}{2}e^{2\Phi}\dot{\sigma}^2-2\Lambda\right]-\Lambda_{\rm M}\right\}, \tag{2}$$

where a dot denotes differentiation with respect to cosmic time, t. The Friedmann constraint derived from Eq. (2) is given by

$$3\dot{\alpha}^2-\dot{\varphi}^2+2\Lambda+\frac{1}{2}\dot{\sigma}^2e^{2\varphi+6\alpha}+\Lambda_{\rm M}e^{\varphi+3\alpha}=0, \tag{3}$$

in terms of the 'shifted' dilaton field, $\varphi \equiv \Phi - 3\alpha$.

A generalization to the spatially flat, Bianchi type I cosmology may also be considered. This effectively results in the introduction of two massless scalar fields into the reduced action (2). These fields parametrize the shear of the models. Similar degrees of freedom also arise when considering the toroidal compactification of higher–dimensional theories. Although we do not consider these extra fields in this paper, their overall contribution to the dynamics can be modelled by introducing a single modulus field, $\dot{\beta}^2 \equiv \sum_i \dot{\beta}_i^2$, into the reduced action (2) [4], and their inclusion could be important in the discussion of chaotic behaviour.

Zero central charge deficit

We first consider the case $\Lambda = 0$ [4]. We assume that $\Lambda_{\rm M} > 0$, and employ the generalized Friedmann constraint equation (3) to eliminate the axion field from the system. The resulting field equations may then be simplified by introducing the new variables and time coordinate[1]:

$$x \equiv \frac{\sqrt{3}\alpha'}{\psi}, \quad z \equiv \frac{\Lambda_{\rm M}}{\psi^2}, \quad \frac{d}{d\Theta} \equiv \frac{1}{\psi}\frac{d}{d\theta} \equiv \frac{1}{\psi}e^{-(\varphi+3\alpha)/2}\frac{d}{dt}, \tag{4}$$

where a prime denotes differentiation with respect to θ and $\psi \equiv \varphi'$. The Friedmann constraint yields

$$1 - x^2 - z \geq 0, \tag{5}$$

from which it follows that the phase space is bounded with

$$0 \leq \{x^2, z\} \leq 1. \tag{6}$$

The invariant set $1 - x^2 - z = 0$ corresponds to a trivial axion field.

The cosmological field equations for the isotropic FRW model can now be expressed in terms of the plane system:

$$\frac{dx}{d\Theta} = (x+\sqrt{3})[1-x^2-z] + \frac{1}{2}z[x-\sqrt{3}] \tag{7}$$

$$\frac{dz}{d\Theta} = 2z\left\{[1-x^2-z] - \frac{1}{2}(1-z-\sqrt{3}x)\right\}. \tag{8}$$

The equilibrium points of this system are $L^+_{(-)}$ $(x, z = -1, 0)$, $L^+_{(+)}$ $(1, 0)$ and S^+ $(-1/3\sqrt{3}, 16/27)$. The first two are saddles and S^+ is a repelling focus. The functional form of these solutions was presented and discussed in Ref. [4] and the phase portrait is given in Fig. 1. We note that the exact solutions corresponding to all of the equilibrium points are self-similar cosmological models [6].

[1] We assume that $\psi > 0$; the case $\psi < 0$ is related to a time-reversal of the system and the qualitative mathematical behaviour is similar (although the physical interpretation is quite different).

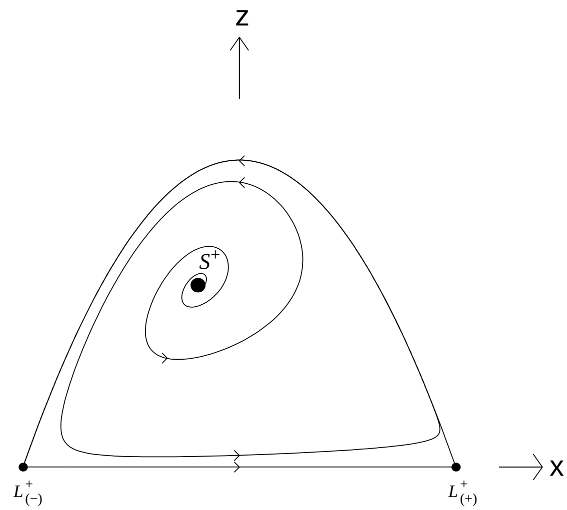

Figure 1: *Phase portrait of the system (7)–(8), corresponding to the isotropic FRW model with* $\Lambda_{\rm M} > 0$ *and* $\Lambda = 0$. *We shall adopt the convention that large black dots represent sources (i.e., repellers), large grey-filled dots represent sinks (i.e., attractors), and small black dots represent saddles. Note that in this phase space orbits are future asymptotic to a heteroclinic cycle.*

We see from Fig. 1 that the orbits are future asymptotic to a *heteroclinic cycle*. This is comprised of the two saddle equilibrium points $L^+_{(-)}$ and $L^+_{(+)}$ and the single (boundary) orbits in the invariant sets $z = 0$ ($\Lambda_{\rm M} = 0$) and $1 - x^2 - z = 0$ ($\dot{\sigma} = 0$) joining $L^+_{(-)}$ and $L^+_{(+)}$. Hence, the orbits exhibit *cyclical* behaviour [4]. For each 'cycle', an orbit is quasi–stationary in the neighbourhood of the saddle point $L^+_{(-)}$. It then shadows the orbit in the invariant set $z = 0$ as it moves rapidly towards the equilibrium point $L^+_{(+)}$. It settles into another quasi–stationary phase close to $L^+_{(+)}$ and eventually moves quickly back to $L^+_{(-)}$ shadowing the orbit in the invariant set $1 - x^2 - z = 0$. It is important to emphasize that the orbits move progressively closer towards the two saddles, $L^+_{(\pm)}$, after the completion of each cycle. Thus, the motion is *not* periodic and a given orbit spends more and more time in the neighbourhood of these equilibrium points.

The physics behind the cyclical nature of these orbits is as follows. The sign of the variable x determines whether the universe is expanding or contracting. The value of this variable passes through zero during each cycle. This behaviour arises because the cosmological constant effectively resists the expansion of the universe, but the axion field has the opposite effect. Since the energy density of the latter scales as $\dot{\sigma}^2 \propto e^{-6\alpha}$, it is negligible when the spatial volume of the universe is large. Consequently, the cosmological constant forces the expanding universe to recollapse. However, the axion field inevitably becomes dominant and reverses this collapse, causing the universe to enter into a new expanding phase. The process is then repeated and the interplay

between the two opposing trends results in a universe that undergoes a series of bounces.

Non-zero central charge deficit

We now consider the case $\Lambda \neq 0$. We again employ Eq. (3) to eliminate the $\dot{\sigma}^2$ term from the field equations, and make the following definitions:

$$x \equiv \frac{\sqrt{3}\dot{\alpha}}{\xi}, \quad y \equiv \frac{-2\Lambda}{\xi^2}, \quad z \equiv \frac{\Lambda_{\rm M} e^{\varphi+3\alpha}}{\xi^2}, \quad u \equiv \frac{\dot{\varphi}}{\xi}, \quad \frac{d}{dt} \equiv \xi \frac{d}{dT}. \tag{9}$$

We assume that $\Lambda < 0$ ($\Lambda_{\rm M} > 0$) and we define $\xi^2 = \dot{\varphi}^2 - 2\Lambda$. The generalized Friedmann constraint equation (3) now yields $0 \le x^2 + z \le 1$, so that all variables are bounded: $0 \le \{x^2, y, z, u^2\} \le 1$. From the definition of ξ, y is given by $u^2 + y = 1$. The resulting three-dimensional system therefore becomes:

$$\frac{dx}{dT} = \sqrt{3}\left(1 - x^2 - \frac{3}{2}z\right) + xu\left(1 - x^2 - \frac{1}{2}z\right), \tag{10}$$

$$\frac{du}{dT} = (1 - u^2)\left(x^2 + \frac{1}{2}z\right) > 0, \tag{11}$$

$$\frac{dz}{dT} = z\left[u\left(1 - 2x^2 - z\right) + \sqrt{3}x\right]. \tag{12}$$

The invariant sets $x^2 + z = 1$, $z = 0$, $u^2 = 1$ define the boundary of the phase space and it is important to note that the variable u is *monotonically increasing*. This ensures that there are no closed or recurrent orbits in the phase space. The equilibrium points of the system are all saddles: $S^{\pm}$ $(x, u, z = \mp 1/\sqrt{27}, \pm 1, 16/27)$, $L^{+}_{(\pm)}$ $(\pm 1, 1, 0)$ and $L^{-}_{(\pm)}$ $(\pm 1, -1, 0)$. The points $L^{+}_{(\pm)}$ represent power–law cosmologies with $\dot{\varphi} > 0$, where only the dilaton field is non–trivial, i.e., the axion field and cosmological constant terms are dynamically negligible. These solutions are termed 'dilaton–vacuum' solutions and have an analytical form given by $e^{\alpha} \propto t^{\pm 1/\sqrt{3}}$ and $e^{\Phi} \propto t^{-1 \pm \sqrt{3}}$. The points $L^{-}_{(\pm)}$ are the corresponding solutions where $\dot{\varphi} < 0$. The phase portrait is given in Fig. 2.

In this case there are *no sinks* and *no sources* in the full three–dimensional phase space. Since the variable u (and hence $\dot{\varphi}$) is monotonically increasing, solutions generically asymptote in both the past and future towards the invariant sets $u = \pm 1$. These both include an heteroclinic cycle and this implies that generically the solutions exhibit similar asymptotic behaviour at both early and late times to that discussed above. (For example, to the future the orbits in the three–dimensional phase space shadow the orbits in the two–dimensional invariant set $\Lambda = 0$.) The orbits interpolate between the dilaton–vacuum solutions corresponding to the saddle points $L^{-}_{(\pm)}$ in the past and the dilaton–vacuum solutions corresponding to the saddle points $L^{+}_{(\pm)}$ in the future. The effect on the dynamics of the cosmological constant, $\Lambda_{\rm M}$, is significant at both early and late times. The points $S^{\pm}$ correspond to the equilibrium point

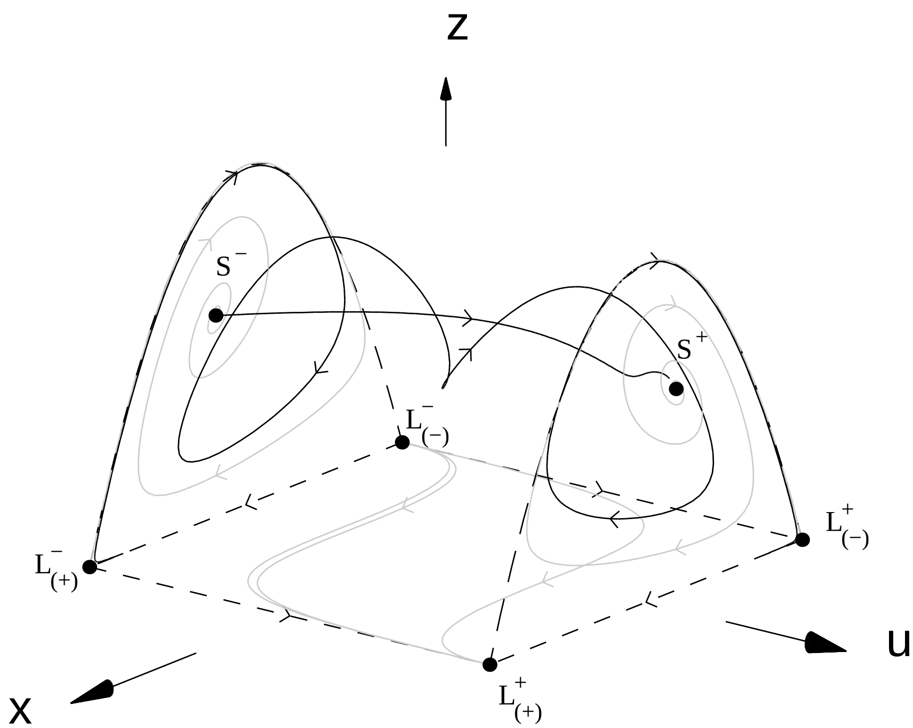

Figure 2: *Phase diagram of the system (10)-(12) for* $\Lambda < 0$ *and* $\Lambda_{\mathrm{M}} > 0$*, where* $\dot{\varphi} > 0$ *is assumed. See caption to Figure 1. Grey lines represent typical trajectories found within the two-dimensional invariant sets, dashed black lines are those trajectories along the intersection of the invariant sets, and solid black lines are typical trajectories within the full three-dimensional phase space.*

S^+ in Fig. 1, but unlike in the $\Lambda = 0$ case in which S^+ is a repelling focus, in the three–dimensional phase space they are saddles and hence they do not play a primary role in the asymptotic behaviour.

From a physical point of view, the cyclical behaviour arises due to the complex interplay between the axion field and the cosmological constant terms. The universe continues to undergo a succession of bounces between expanding and contracting phases due to the axion field and vacuum energy $\Lambda_{\rm M}$. However, the inclusion of the central charge deficit, Λ, causes $\dot{\varphi}$ to ultimately change sign. Thus, the asymptotic behaviour in the future is related to a time–reversal of the asymptotic behaviour in the past. We remark that the dilaton–vacuum solution, $e^\alpha \propto t^{-1/\sqrt{3}}$, corresponding to the point $L^+_{(+)}$ is inflationary over the range $t < 0$, because the expansion is accelerating. In this case, the accelaration is driven by the kinetic energy of the dilaton field. (For a recent review of the cosmological significance of these solutions see, e.g., Ref. [7]).

Finally, we make some brief remarks on the case $\Lambda > 0$. We can define $\xi \equiv \dot{\varphi}$ and consider the subset $\dot{\varphi} \geq 0$. Introducing normalized variables as before yields a three–dimensional, compact system of autonomous, ordinary differential equations. We have completed a full dynamical analysis of this system, but we only describe the main features here. There is a non-hyperbolic equilibrium point, C^+, which can be shown to be a (global) source, since y is a monotonically decreasing function. This point represents a static universe, where the dilaton field is evolving linearly with time and the axion field and $\Lambda_{\rm M}$ are dynamically negligible. There are also two saddle points, $L^+_{(\pm)}$, which represent dilaton–vacuum solutions; these are analogues of the saddles that appear above. Again, there is also a saddle S^+. We stress that there are no sinks in the phase space. Therefore, trajectories generically asymptote into the past towards C^+, and then spiral away towards the heteroclinic cycle in the invariant set $y = 0$ containing the saddle points $L^+_{(-)}$ and $L^+_{(+)}$. The phase space is depicted in Fig. 3.

3 Discussion

The most important mathematical feature of the models we have considered is the *cyclical* behaviour that arises due to the existence of a heteroclinic cycle. This is of great physical significance, because it might be an indicator of possible chaotic behaviour. The solutions interpolate between different 'Kasner-like' dilaton–vacuum, power–law models, undertaking cycles between the saddles in the three–dimensional phase space. This is similar to the dynamical behaviour that occurs in spatially homogeneous Bianchi cosmological models [6, 8]. The question of chaos in anisotropic Bianchi type IX string cosmologies has been considered [9]. It was shown that since the axion and dilaton fields behave collectively as a stiff perfect fluid, the system oscillates only a finite number of times. Consequently, there is no Mixmaster-type chaos in these models [8, 10]. This is to be expected since it is known that the admission of stiff

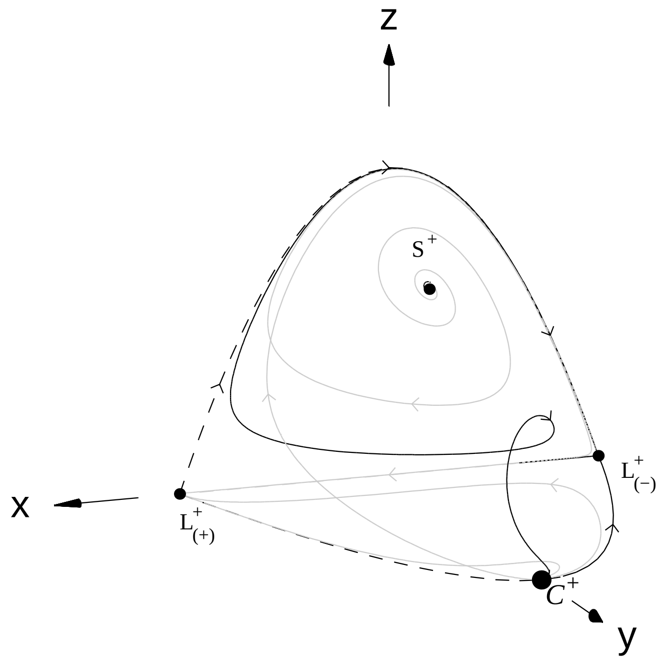

Figure 3: *Phase diagram of the system with* $\Lambda > 0$ *and* $\Lambda_{\rm M} > 0$*, where* $\dot{\varphi} > 0$ *is assumed. See captions to Figures 1 and 2.*

fluid matter causes chaos to cease [11]. In contrast to the anisotropic Bianchi type IX cosmologies, however, the models described above contain cosmological terms, e.g., an effective dilaton potential and a dynamically important axion field. It is these intrinsically stringy effects that give rise to chaotic behaviour and this chaos has a different origin to the chaotic behaviour that arises in general relativistic models. On the other hand, there may be some connection with models that contain Yang–Mills fields. It is known that chaotic oscillations occur for such fields [12] and, moreover, it was shown in [11] that the oscillations that are suppressed by a single massless scalar field can be restored by coupling an electromagnetic field to a Brans-Dicke type field. This model is related to a scalar field model with an exponential potential [13] and, consequently, is also related to string theory cosmological models [14].

There are a number of outstanding issues that need to be addressed regarding the possible existence of chaos in string cosmology. First, the chaotic behaviour depends crucially on the dimensionality of spacetime and on the product manifold structure of the extra dimensions [9]. In particular, superstring theories are formulated in $D = 10$ spacetime dimensions, while M-theory, with its low-energy supergravity limit, is an eleven–dimensional theory [15]. Second, the low energy effective action is only valid in the perturbative regime of weak coupling and small curvature. In general, it may be necessary to study chaos within the context of a full non-perturbative formulation of the theory, but at present such a formulation is unknown. Nevertheless, if chaotic

behaviour occurs at the level of the effective action, it is to be expected that similar behaviour should arise in the non–perturbative regime. Finally, there is the question of what will happen if inhomogeneities are introduced. Again, such effects will be most unlikely to lead to any suppression of chaotic behaviour and will perhaps make chaos even more predominant [9].

There are other questions which are important in early universe cosmology in general, and in string cosmology in particular. The questions of whether cosmological models can isotropize and/or inflate (and if they can inflate whether there is a graceful exit from inflation) are of great importance [16]. The techniques utilized in this paper can easily be adapted to study the possible isotropization in more general spatially homogeneous but anisoptropic string cosmological models. Inflationary properties of simple string cosmologies have been discussed above. However, a chaotic cosmological regime might either be an alternative to inflation or, perhaps more importantly, could work in tandem with an inflationary mechanism [17] to produce new interesting physical phenomena. For example, a chaotic regime due to dissipative effects or chaotic mixing [18] could possibly be an alternative to inflation as a cause of homogenization and isotropization. This might alleviate the problems of initial conditions in inflation. This last point has been addressed in [19], where it was suggested that there would be sufficient time for a compact, negatively–curved universe to homogenize since chaotic mixing smooths out primordial fluctuations in a pre-inflationary period.

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